Math 112: Introductory Real Analysis

§ Lecture 1 (Jan 27, 2025)

The real field $\mathbb{R}$ plays a fundamental role in analysis.

The rational number system $\mathbb{Q}$ is inadequate for many purposes.

Example

$p^2 = 2$ is not satisfied by any rational p.

proof) If there were such p, we can write $p = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ that are not both even.

Then, $m^2 = 2n^2$, which means m^2 is even, and hence so is m.

That implies $2n^2$ is divisible by 4, so that n is even,

which is a contradiction to our choice of m and n.

Therefore, there's no rational p that satisfies $p^2 = 2$. ■

Let $A = \{p \in \mathbb{Q} \mid p > 0,\ p^2 < 2\}$ and $B = \{p \in \mathbb{Q} \mid p > 0,\ p^2 > 2\}$

Then A contains no largest number and B contains no smallest

There are "gaps" in rational number system!

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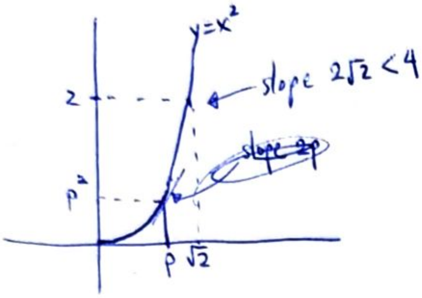
proof that A contains no largest number and B contains no smallest)

For every $p \in A$, we need to find $q \in A$ such that $p < q$



Can take $q = p + \dfrac{2-p^2}{4}$

Then, $2-q^2 =$

$$= 2 - \left(p + \frac{2-p^2}{4}\right)^2$$

$$= (2-p^2) - \frac{p}{2}(2-p^2) - \frac{1}{16}(2-p^2)^2$$

$$= (2-p^2)\left(1 - \frac{p}{2} - \frac{1}{16}(2-p^2)\right)$$

$$> (2-p^2)\left(\frac{7}{8} - \frac{p}{2}\right) > 0$$

$$\left(\frac{7}{4} > p\right)$$

Hence, A contains no largest number.

(A more slick way to prove is to pick $q = p + \dfrac{2-p^2}{p+2} = \dfrac{2p+2}{p+2}$

so that $2 - q^2 = \dfrac{2(2-p^2)}{(p+2)^2}$,

which works for both A and B.) ∎

In order to elucidate the general structure of real numbers, let's start with the discussion of <u>ordered sets</u> and <u>fields</u>.

<u>Def</u> Let S be a set.

An <u>order</u> on S is a relation, denoted by $<$, with the following two properties:

(1) If $x \in S$ and $y \in S$, then one and only one of the statements

$$x < y, \quad x = y, \quad y < x$$

is true.

(2) (transitivity) For $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

(Notation: $y > x$ means $x < y$.
$x \le y$ means $x < y$ or $x = y$.)

<u>Def</u> An <u>ordered set</u> is a set S with an order $<$ on it.

<u>E.g.</u> $\mathbb{Q}$ is an ordered set, if $r < s$ is defined to mean $s - r > 0$
(i.e. the usual ordering)

<u>Def</u> Suppose S is an ordered set, and $E \subseteq S$.
If $\beta \in S$ is such that $x \le \beta$ for every $x \in E$,
we say β is an <u>upper bound</u> of E.
When E has an upper bound, it is said to be <u>bounded above</u>
Lower bounds are defined in the same way (with $\geq$ in place of $\le$)

<u>Def</u> α is called the <u>least upper bound</u> or <u>supremum</u> of $E \subseteq S$

if it is the smallest upper bound, i.e.

(i) α is an upperbound of E,

(ii) If $\gamma < \alpha$, then γ is not an upperbound of E.

In that case, we write $\alpha = \sup E$.

Likewise, the <u>greatest lower bound</u>, or <u>infimum</u>, of E

is defined in a similar manner, and is denoted by $\inf E$.

<u>Examples</u>

- $A = \{p \in \mathbb{Q} \mid p > 0, p^2 < 2\}$ has no least upperbound in $\mathbb{Q}$

and $B = \{p \in \mathbb{Q} \mid p > 0, p^2 > 2\}$ has no greatest lower bound in $\mathbb{Q}$.

- $E = \{\frac{1}{n} \mid n = 1, 2, 3, \cdots\} \Rightarrow$ $\sup E = 1$
  $\inf E = 0$ ← not in E!

<u>Def</u> An ordered set S is said to have the <u>least-upper-bound property</u>

if for any non-empty, bounded above subset $E \subseteq S$,

$\sup E$ exists in S.

E.g. $\mathbb{Q}$ does NOT have the least upperbound property

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<u>Thm</u> Suppose S is an ordered set with the least-upper-bound property,

$B \subseteq S$ nonempty, bounded below.

Let $L = \{x \in S \mid x \text{ is a lower bound of } B\}$.

Then, $\sup L$ and $\inf B$ exists in S, and

$$\sup L = \inf B.$$

In particular, <u>every ordered set with the least-upper-bound property also has the greatest-lower-bound property</u>.

proof) L is non-empty (since B is bounded below)

and bounded above (since B is non-empty).

Therefore, $\sup L$ exists in S.

$\sup L \leq x$ for every $x \in B$ (since, if $x < \sup L$, then x is not an upper bound of L, meaning $x \notin B$)

Thus $\sup L \in L$.

Moreover, if $\sup L < x$, then $x \notin L$ (since $\sup L$ is an upper bound of L).

This means that $\sup L$ is the greatest lower bound of B.

(i.e. the largest element of L)

i.e. $\sup L = \inf B$. ∎